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Note

A Note on a Result of Bernstein

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According to S. N. Bernstein [1, p. 90], for any $n \geqslant 0$, the error in best uniform approximation of $(1-x)^{-1}$ on $[-\frac{1}{2}, \frac{1}{2}]$ by polynomials of degree $\leqslant n$ having integral coefficients and a leading coefficient 1 is 2^{-n} and is realized by $\sum_{i=0}^{n} x^i$. In this note we establish among other things that $(1-x)^{-1}$ can be approximated on $[-\frac{1}{2}, \frac{1}{2}]$ by polynomials $P_{n-1}(x)$ of degree $n-1$ $(n \geqslant 1)$ with an error $2[T_n(2)]^{-1} < 4(2+\sqrt{3})^{-n}$. Here $T_n(x)$ denotes the Chebyshev polynomial of the first kind of degree n. Further we establish that the error obtained in approximating $(1-x)^{-1}$ on $[-\frac{1}{2}, \frac{1}{2}]$ by polynomials of degree $\leqslant n-1$ is never smaller than $\frac{2}{3}[T_n(2)]^{-1}$. Thus the error of best approximation is $C_n[T_n(2)]^{-1}$ with a bounded C_n. It is a special case of a more general result we obtain. We also note that for any constant $a \geqslant 2$, $(1-x)^{-1}$ can be approximated on $[-1/a, 1/a]$ by polynomials of degree n, having non-negative, nonincreasing coefficients only with an error $< a(a^{n+1}-1)^{-1}$, but never better than $(a^{n+1}-1)^{-1}$. We also show for $n \geqslant 0$, that the smallest maximal error in a uniform approximation of $1-x$ on $[0, 1]$ by ratios of polynomials of degree $\leqslant n$, having nonnegative, non-increasing coefficients is $(n+2)^{-1}$.

THEOREM 1.

$$\left\| (1-x) - \frac{n+1}{(n+2)\sum_{i=0}^{n} x^i} \right\|_{L^{\infty}_{[0,1]}} \leqslant \frac{1}{n+2}, \qquad n = 0, 1, 2, \ldots. \tag{1}$$

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Proof. For $0 \leqslant x \leqslant (n+2)^{-1/(n+1)}$,

$$(1-x)\sum_{i=0}^{n} x^i = 1 - x^{n+1} \geqslant 1 - \frac{1}{n+2} = \frac{n+1}{n+2},$$

$$0 \leqslant (1-x) - \frac{n+1}{(n+2)\sum_{i=0}^{n} x^i} = \frac{1 - x^{n+1} - [(n+1)/(n+2)]}{\sum_{i=0}^{n} x^i}$$

$$= \frac{(n+2)^{-1} - x^{n+1}}{\sum_{i=0}^{n} x^i} \leqslant \frac{1}{n+2}.$$

For $(n+2)^{-1/n+1} < x \leqslant 1$,

$$0 < \frac{n+1}{(n+2)\sum_{i=0}^{n} x^i} - (1-x) = \frac{-(n+2)^{-1} + x^{n+1}}{\sum_{i=0}^{n} x^i} = g(x);$$

and as $g'(x) > 0$ in $[0, \infty]$, $g(x) \leqslant g(1) = 1/(n+2)$, proving (1).

THEOREM 2. *Let $P(x)$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$, $b_0 > 0$, be real polynomials of degree $\leqslant n$ $(n \geqslant 0)$ having nonnegative, nonincreasing coefficients. Then*

$$\left\| 1 - x - \frac{P(x)}{Q(x)} \right\|_{L^{\infty}_{[0,1]}} \geqslant \frac{1}{n+2}. \tag{2}$$

Proof. Set

$$\left\| 1 - x - \frac{P(x)}{Q(x)} \right\|_{L^{\infty}_{[0,1]}} = \varepsilon.$$

Then on $[0, 1]$, with $P(x) = \sum_{i=0}^{n} a_i x^i$, we have

$$\varepsilon \geqslant \frac{P(x)}{Q(x)} - (1-x) \geqslant \frac{a_0}{b_0 \sum_{i=0}^{n} x^i} - (1-x)$$

$$= \left(\frac{a_0}{b_0} - 1\right)\frac{1}{\sum_{i=0}^{n} x^i} + \frac{1}{\sum_{i=0}^{n} x^i} - (1-x)$$

$$\geqslant \frac{-\varepsilon}{\sum_{i=0}^{n} x^i} + \frac{1}{\sum_{i=0}^{n} x^i} - (1-x),$$

$$\varepsilon\left(1 + \sum_{i=0}^{n} x^i\right) \geqslant 1 - (1-x)\sum_{i=0}^{n} x^i = x^{n+1},$$

$$\varepsilon \geqslant \frac{x^{n+1}}{1 + \sum_{i=0}^{n} x^i}.$$

Thus,

$$\varepsilon \geqslant \max_{0 \leqslant x \leqslant 1} \frac{x^{n+1}}{1+\sum_{i=0}^{n} x^i} \geqslant \frac{1}{n+2}.$$

Theorems 1 and 2 imply that for $n=0, 1, 2, \ldots$, the smallest value of the left-hand side of (2) subject to the above conditions on $P(x)$ and $Q(x)$ is $1/(n+2)$ and that (1) holds with equality sign.

THEOREM 3. *For any constant $a \geqslant 2$ and $x \in [-1/a, 1/a]$, we have for $n=0, 1, 2, \ldots$,*

$$0 \leqslant \left(\frac{a^{n+1}}{a^{n+1}-1}\right) \sum_{i=0}^{n} x^i - \frac{1}{1-x} < \frac{a}{a^{n+1}-1}.$$

Proof.

$$ax-(ax)^{n+1} = ax(1-(ax)^n) < a-1,$$

and hence the result.

Remarks. (1) Actually $a \geqslant 1+n(n+1)^{-(n+1)/n}$, will be enough for our theorem.

(2) Let $P(x)$ and $Q(x)$ satisfy the assumptions of Theorem 2 and $a \geqslant 2$. Then by adopting the technique used in the proof of Theorem 2, we obtain

$$\left\| \frac{1}{1-x} - \frac{P(x)}{Q(x)} \right\|_{L^\infty[-1/a, 1/a]} \geqslant \frac{1}{a^{n+1}-1}.$$

THEOREM 4. *Let $a>1$ and $n>1$. Then there exists a polynomial $P_{n-1}(x)$ of degree $n-1$ such that*

$$\left\| \frac{1}{1-x} - P_{n-1}(x) \right\|_{L^\infty[-1/a, 1/a]} \leqslant \frac{a}{(a-1)\, T_n(a)}.$$

Proof. Set

$$P_{n-1}(x) = \frac{T_n(a)-T_n(ax)}{(1-x)\, T_n(a)}.$$

If $x \in [-1/a, 1/a]$, then

$$\left|\frac{1}{1-x} - P_{n-1}(x)\right| = \left|\frac{T_n(ax)}{(1-x)\,T_n(a)}\right| \leqslant \frac{a}{(a-1)\,T_n(a)}.$$

THEOREM 5. *Let $a > 1$, $n \geqslant 1$, and let $P_{n-1}(x)$ be a polynomial of degree $\leqslant n-1$. Then*

$$\left\|\frac{1}{1-x} - P_{n-1}(x)\right\|_{L^\infty[-1/a,\,1/a]} \geqslant \frac{a}{(a+1)\,T_n(a)}.$$

We need the following [1, p. 68]

LEMMA. *If a polynomial $P_n(x)$ of degree $\leqslant n$, $n \geqslant 0$, satisfies the inequality $|P_n(x)| \leqslant L$ on $[c, d]$, then at any real point x outside $[c, d]$ we have*

$$|P_n(x)| \leqslant L \left|T_n\left(\frac{2x-c-d}{d-c}\right)\right|.$$

Proof of Theorem 5.

Let $P_{n-1}(x)$ deviate least from $(1-x)^{-1}$ on $[-1/a, 1/a]$ in the uniform norm. Set

$$\left\|\frac{1}{1-x} - P_{n-1}(x)\right\|_{L^\infty[-1/a,\,1/a]} = \varepsilon, \tag{3}$$

$$P(x) = 1 - (1-x)\,P_{n-1}(x). \tag{4}$$

From (4) and (3) we obtain

$$\max_{[-1/a,\,1/a]} |P(x)| \leqslant \frac{a+1}{a}\,\varepsilon.$$

By the lemma,

$$1 = |P(1)| \leqslant \frac{a+1}{a}\,\varepsilon T_n(a),$$

proving our theorem.

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REFERENCE

1. A. F. TIMAN, "Theory of Approximation of Functions of a real Variable," Macmillan Co., New York, 1963.